Non-Linear Vibration Analysis of Mechanical Structure System Using Substructure Synthesis Method

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(Received February 25, 1999)

In this paper, a new method of analysis of a complex nonlinear vibration system is presented. Combining substructure synthesis and perturbation methods the computation cost for large mechanical system is considerably reduced.

The system is divided into components. Under the assumption that the mode shape does not change, these are approximately transformed to modal coordinate system with nonlinearity. Using perturbation method, the overall modal equations are derived and solved sequentially. These solutions are synthesized to the overall system. Solution of the overall system is obtained in the modal coordinate system and then, these are translated into the physical coordinate. This method is applied to a vibration analysis of a large mechanical structure system composed of the rotor-bearing and casing. In order to illustrate accuracy and computation time of the proposed method, the results are compared with those obtained by the finite element method.

Key Words : Vibration of Rotor System, Substructure Synthesis Method, Non-Linear Vibration, Response Analysis-

1. Introduction

In structural dynamics, numerical integration of equation of motion is used frequently after formulating the structures by finite element method in order to find the response of structure. These methods assume that the objective system is a linear one. But in the actual systems, the response characteristics, experimentally obtained, are generally affected by the nonlinearity. Therefore, for accurate structural dynamic analysis of the real system, nonlinear characteristics had to be considered. Dynamic response of multi-degrees of freedom (MDOF) nonlinear structure is usually determined by the numerical integration of motion equation. This nonlinear dynamic analysis is time consuming and expensive, particularly if the computations had to be repeated several times to arrive at representative response values for design. Therefore, in many branches of structural dynamics, approximate analytical techniques are used as an alternative to numerical integration procedures for the steady state response analysis of nonlinear MDOF structures. A number of analytical methods for nonlinear systems have been investigated. These include the averaging method (Glisinn, 1982), the Ritz-Galerkin method (Ymaki, 1980), perturbation method (Harris and Crede, 1987) and harmonic balance technique (Choi and Noah, 1987). Nataraj and Nelson (Nataraj and Nelson, 1989) used trigonometric collocation to analyze the periodic response of rotor dynamic systems which is affected by the nonlinearities.

In recent years, reduction of computation time for a given level of accuracy in the case of linear and nonlinear dynamic problems has become the focus of intense research efforts. The dynamic analysis of structural systems requires the solution

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of large order sets of linearized differential equations of motion. These linear equations are frequently diagonalized by means of an eigenvector transformation, which greatly simplifies the analysis. Such large order systems may be time consuming to set up and costly to solve in terms of computer time and storage. The sub-structure synthesis method developed herein allows for significant reduction in the size of the overall system problem which retaining the essential dynamic characteristic.

The authors (Iwatsube, et. al., 1998) proposed a new method to analyze the dynamic problems of MDOF systems with nonlinearity by using substructure synthesis method (SSM). The SSM technique can reduce the overall size of the problem for the nonlinear structure. The approximate solution of nonlinear rotor system is obtained using the perturbation method. However, in rotating machinery, ball bearing clearances, squeeze-film dampers, journal bearings, seals, and frictional forces, etc. contribute to the nonlinearity. When such system is subjected to a periodic external excitation, they respond with a variety of complex dynamic behaviors in certain parameter ranges.

Therefore, this paper developed more expanded analytical technique of the nonlinear rotating machinery vibration including bearing nonlinearity by applying the SSM using the normal modes of the vibration, which are obtained by linear part of equation of motion, and the perturbation method. This method is applied to rotor system in order to demonstrate the performance of the method in respect to accuracy and computation time. Results are investigated for the effect of bearing and rotor nonlinearity to the overall system.

2. Method of Analysis

The structural system to be considered consists of a set of interconnecting components that are segments with distributed mass and elasticity and, in general, nonlinear parts. We will assume that the structure is divided into linear and nonlinear substructures with nonlinear conjunction region in the nonlinear structural problems as shown, in Fig. 1. First stage in the process is the sub-structuring of the original nonlinear system into components that can be modeled separately with linear and nonlinear set. This step results in the division of a larger system into smaller subsystems, which may be easier to model, and enables the segregation of linear and nonlinear components.

2.1 Substructuring the overall system

Assuming that the overall dynamic system is divided into 2 components, the first component of internal region with nonlinearity is now considered as substructure 1 for the analysis. The second component of internal region without nonlinearity is considered as substructure 2. And there is an elastic assembling region between both components. A complete formulation of the analysis is presented regarding that the model is the rotor system as a case of nonlinear structure system.

The nonlinear components include the shaft

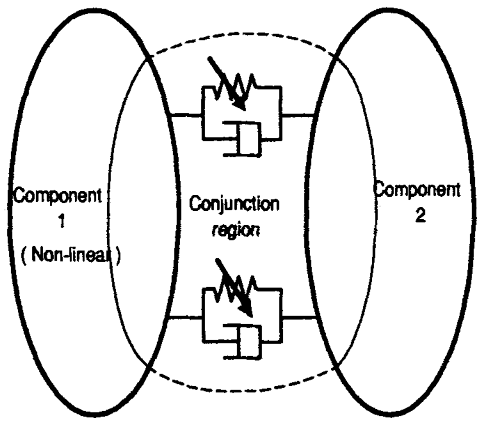

Fig. 1 Sub-structuring of nonlinear system.

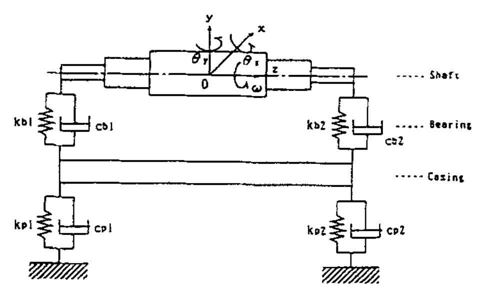

Fig. 2 Rotor-bearing-casing system.

and the bearing as interconnection region. Pedestal is the linear component. Coordinate systems of rotor are shown in Fig. 2. The o-xyz coordinate system is fixed in space such that the origin coincides with the center of the shaft, x-axis is horizontal of shaft, y-axis is vertically upwards. The acceleration of gravity and any non-uniformity in the cross-section along its length are ignored.

2.2 Component 1-Shaft with nonlinearity

The shaft is modeled by using finite element method. At that time, the nonlinear restoring force characteristic of each element is considered as a relation that the stress of the element by bending moment is equalized with the restoring force which is cubic stiffness type nonlinearities as follows:

$$\sigma=E(\bar{\varepsilon}+\gamma\cdot\bar{\varepsilon}^3) \tag{1}$$

where σ, $\bar{\varepsilon}$ and γ denote the stress, strain and the material coefficient, respectively. To determine the strain energy accompanying the deformation of elastic rotor due to bending, let us consider the strain-stress relationship of Eq. (1) disregarding axial deformation. The nonlinear stiffness term for the rotor element is determined approximately by a similar way with linear stiffness matrix by considering the strain energy. Thus, the nonlinear restoring force term can be formulated in complex form including a cubic term. As the first step of analysis method, we assume the nonlinear restoring term $[R]$ can be approximated in a simple form as follows:

$$[R]=[K_1]\{X_1\}+\varepsilon[X_N]\{X_1^3\} \tag{2}$$

where $[K_1]$, $[K_N]$ are the stiffness matrix and nonlinear stiffness matrix, respectively. By using the finite element method, adding the external force and considering the boundary conditions, the equation of motion for the rotor as component 1 can be written as follows:

$$[M_1]\{\ddot{X}_1\}+[K_1]\{X_1\}+\varepsilon[K_N]\{X_1^3\} =\{F_1\}+\{f_{b1}\} \tag{3}$$

where $[M_1]$ is the mass matrix. $\{F_1\}$ and $\{f_{b1}\}$ are exciting force vector and internal force vector, respectively. ε is a small parameter which is defined in terms of γ. The displacement vector $\{X_1\}=\{x_m, y_m, \theta_{xm}, \theta_{ym}\}^T$, $(m=1, 2, \cdots, J)$, is consist of the displacements and rotations for the x and y-directions for the m-th nodal point. J is the number of elements. In order to obtain the modal coordinate system, the linear homogeneous equation of Eq. (3) is analyzed where the governing vibration mode of the system is assumed to be linear in spite of nonlinearity. By using the modal matrix $[\Phi_1]$, the displacement $\{X_1\}$ in physical coordinate can be transformed into the displacement $\{\xi_1\}$ in the modal coordinate as follows:

$$\{X_1\}\equiv[\Phi_1]\{\xi_1\} \tag{4}$$

Here, subscript denotes substructure 1. Substituting Eq. (4) into Eq. (3) and pre-multiplying by $[\Phi_1]^T$, Eq. (3) can be expanded to the modal equations, which consist of the equations with respect to the linear and nonlinear term.

$$[\backslash I\backslash]\{\ddot{\xi}_1\}+[\backslash\omega_1^2\backslash]\{\xi_1\}+\varepsilon[\Phi_1]^T[K_N]\{X_1^3\} =\varepsilon\{f_{\varepsilon 1}\}+f_{b1} \tag{5}$$

where $\{f_{\varepsilon 1}\}=[\Phi_1]^T\{F_1\}$. f_{b1} is the internal force term in modal coordinates. Usually $[\Phi_1]^T[K_N]\{X_1^3\}$ is not diagonal matrix. But as $[K_1]$ and $[K_N]$ are derived in almost similar processes, this term will be changed in modal coordinates as follows:

$$\{X_1^3\}=\begin{Bmatrix} x_{11}^3 \\ x_{12}^3 \\ \vdots \\ x_{1n}^3 \end{Bmatrix}=\begin{Bmatrix} [\phi_{11}\xi_{11}+\phi_{12}\xi_{12}+\cdots+\phi_{1n}\xi_{1n}]^3 \\ [\phi_{21}\xi_{11}+\phi_{22}\xi_{22}+\cdots+\phi_{2n}\xi_{2n}]^3 \\ \vdots \\ [\phi_{n1}\xi_{11}+\phi_{n2}\xi_{n2}+\cdots+\phi_{nn}\xi_{nn}]^3 \end{Bmatrix} \tag{6}$$

In Eq. (6), $(\xi_{11}, \cdots, \xi_{1n})$ are coupled each other. By considering their ratio Eq. (6) can be treated in a compact form. To obtain the ratio of each modal coordinate displacement, the eigenvector matrix $[\Phi_T]$ of overall system with only linear part is analyzed. The following transformation with $[\Phi_T]$ is applied.

$$\{\xi\}=[\Phi_T]\{\eta\} \tag{7}$$

The first mode is a governing mode of the vibration of the system comparing with the other higher modes because the system is excited around the 1st natural frequency. Therefore, we use only η_1 with neglecting the higher order to obtain modal coordinates ratio approximately.

$$\xi_{12}=\frac{\Phi_{T21}}{\Phi_{T11}}\xi_{11},\ \xi_{13}=\frac{\Phi_{T31}}{\Phi_{T11}}\xi_{11}\ \cdots,$$
$$\xi_{1n}=\frac{\Phi_{Tn1}}{\Phi_{T11}}\xi_{11} \tag{8}$$

By using the Eq. (8), the nonlinear term of the Eq. (5) can be transformed into modal coordinates.

$$[\Phi_T]^T[K_N]\{X_1^3\}=[\Phi_1]^T[K_N][\Phi_N]\{\xi_1^3\} =[\backslash\omega_{1N}^2\backslash]\{\xi_1^3\} \tag{9}$$

where each component of $[\Phi_N]$ are as follows:

$$\Phi_{Ni1}=\phi_{i1}^3+3\phi_{i1}^2\left(\phi_{i2}\frac{\Phi_{Z21}}{\Phi_{Z11}}+\phi_{i3}\frac{\Phi_{Z31}}{\Phi_{Z11}}+\cdots+\phi_{in}\frac{\Phi_{Zn1}}{\Phi_{Z11}}\right)+\left(6\phi_{i1}\phi_{i2}\phi_{i3}\frac{\Phi_{Z11}\Phi_{Z21}\Phi_{Z31}}{\Phi_{11}^3}+\cdots+6\phi_{in-2}\phi_{in-1}\phi_{in}\frac{\Phi_{n-21}\Phi_{n-11}\Phi_{n1}}{\Phi_{11}^3}\right)$$
$$\Phi_{Nir}=\phi_{ir}^3+3\phi_{ir}^2\left(\phi_{i1}\frac{\Phi_{Z11}}{\Phi_{Zr1}}+\phi_{i2}\frac{\Phi_{Z21}}{\Phi_{Zr1}}+\cdots+\phi_{in}\frac{\Phi_{Zn1}}{\Phi_{Zr1}}\right) \tag{10}$$

$(i=1\sim n,\ r=2\sim n)$. Thus, the nonlinear term is approximated as diagonal matrix resulting in efficient analysis. Therefore, all components of Eq. (3) become diagonal matrix. i th element of Eq. (3) is written as follows under the assumption of proportional weak damping and external force:

$$\ddot{\xi}_{1i}+\varepsilon 2\zeta_{1i}\omega_{1i}\dot{\xi}_{1i}+\omega_{1i}^2\xi_{1i}+\varepsilon\omega_{1Ni}^2\xi_{1i}^3 =f_{\xi 1i}+f_{b1},\ (i=1,2,\cdots,n) \tag{11}$$

n is the number of modes. In Eq. (11), the small variant $\varepsilon\omega_{1ni}^2$ can be regarded as the perturbation parameter term, because the variant $\varepsilon\omega_{1ni}^2$ is relatively smaller than ω_{1i}^2. The dynamic responses ξ_{1i} can be expanded in terms of a series of the perturbation parameter ε expressed as follows:

$$\xi_{1i}=\xi_{1i}^{(0)}+\varepsilon\xi_{1i}^{(1)}+\varepsilon^2\xi_{1i}^{(2)}+\cdots, \tag{12}$$
$$\omega_{1ni}^2=\omega_{1i}^2+\varepsilon\mu_{1i}^{(1)}+\varepsilon^2\mu_{1i}^{(2)}+\cdots, \tag{13}$$

where ω_{1ni}^2 is the frequency of the nonlinear oscillations. Here, superscripts $(^{(0),(1),(2)})$ denote the perturbation order. Substituting Eqs. (12) and (13) into Eq. (11), rearranging them and neglecting the terms involving ε^2, ε^3 and ε^4, we obtain the following equations

$$\ddot{\xi}_{1i}^{(0)}+\omega_{1ni}^2\xi_{1i}^{(0)}=f_{\xi 1i}+f_{b1}^{(0)} \tag{14}$$

$$\ddot{\xi}_{1i}^{(1)}+\omega_{1ni}^2\xi_{1i}^{(1)}=f_{p1}(\xi_{1i}^{(0)},\dot{\xi}_{1i}^{(0)})+f_{b1}^{(1)} \tag{15}$$

where $f_{b1}^{(0)}$, $f_{b1}^{(1)}$ are perturbed internal force terms. The term f_{p1} is the displacement dependent nonlinear term.

$$f_{p1}(\xi_{1i}^{(0)},\dot{\xi}_{1i}^{(0)})=\mu_{1i}^{(1)}\xi_{1i}^{(0)}-\omega_{1Ni}^2\xi_{1i}^{(0)3}-2\zeta_{1i}\omega_{1i}\dot{\xi}_{1i}^{(0)} \tag{16}$$

Although the above procedure is theoretically valid for equations of any order, the vibration modes of each subsystem are limited owing to its restriction. It is clear in rotating machinery such as aircraft engine, the amplitude of vibration modes of the casing is smaller comparing with the shaft one. Therefore, it is assumed that the internal force of subsystem is not so effective to the shaft vibration. Inserting the perturbation 0th order solution of Eq. (14) without internal force into the perturbation 1st order of Eq. (15) and using the trigonometric relation, we obtain the developed Eq. (15). Then, the particular solution of Eq. (15) can be obtained. ω_{1ni}^2 are derived from the relations of perturbed frequency of Eq. (13). In the numerical analysis, Eqs. (14) and (15) can be solved simultaneously, so nonlinear modal Eq. (11) can be written by simultaneous linear equations which variables are $\{\xi_{1i}^{(0)},\xi_{1i}^{(1)}\}^T$.

2.3 Component 2-Casing

To apply the substructure synthesis method, the nonlinear rotor system is divided into three components, which are the nonlinear rotor, casing and bearing as conjunction region.

The casing, which is the linear system, is also modeled by using finite element method. The equation of motion can be written as

$$[M_2]\{\ddot{X}_2\}+[K_2]\{X_2\}=\{F_2\}+\{f_{b2}\} \tag{17}$$

where $[M_2]$ and $[K_2]$ are the mass matrix and stiffness matrix. $\{F\}$, $\{f_{b2}\}$ are exciting force vector and internal force vector, respectively. After the eigenvalue analysis, the Eq. (17) is changed into modal coordinate as the same way as in component 1 under the assumption of proportional weak damping,

$$\ddot{\xi}_{2i}+\varepsilon 2\zeta_{2i}\omega_{2i}\dot{\xi}_{2i}+\omega_{2i}^2\xi_{2i}=f_{\xi 2i}+f_{b2} \tag{18}$$

where $\{f_{\xi 2i}\}=[\Phi_2]^T\{F_2\}$. f_{b2} is internal force term in modal coordinates.

$$\begin{aligned} &\ddot{\xi}_{2i}^{(0)} + \omega_{2i}^2 \xi_{2i}^{(0)} = f_{\xi 2i} + f_{b2}^{(0)} \\ &\ddot{\xi}_{2i}^{(1)} + \omega_{2i}^2 \xi_{2i}^{(1)} = f_{p2}(\dot{\xi}_{2i}^{(0)}) + f_{b2}^{(1)} \end{aligned} \tag{19}$$

where $f_{b2}^{(0)}$, $f_{b2}^{(1)}$ are perturbed internal force term in modal coordinates.

2.4 Assembling region-Bearing with nonlinearity

To apply the substructure synthesis method, the nonlinear bearing is considered as assembling region. The nonlinear bearings are modeled as ball bearings in this case, such that they have a cubic nonlinear term, where the force and displacement expressions are given in matrix form. Generally, there is a damping term in the bearing, but it is ignored in this study. Displacement of the rotor due to the bearing displacements are given by X_b.

$$[K_b]\{X_b\} \pm \delta\{N_b\} = \{F_b\} \tag{20}$$

where $[K_b]$ is the bearing stiffness matrix and $\{F_b\}$ is bearing force. $\delta\{N_b\}$ is a displacement-dependent nonlinear term of the bearing where δ is defined in terms of nonlinear characteristic.

$$\begin{bmatrix} k_{b11} & k_{b12} \\ k_{b21} & k_{b22} \end{bmatrix} \begin{Bmatrix} x_{b1} \\ x_{b2} \end{Bmatrix} + \varepsilon \begin{bmatrix} k_{B11} & k_{B12} \\ k_{B21} & k_{B22} \end{bmatrix} \begin{Bmatrix} x_{b1}^3 \\ x_{b2}^3 \end{Bmatrix} = \begin{Bmatrix} f_{b1} \\ f_{b2} \end{Bmatrix} \tag{21}$$

where f_{b1}, f_{b2} are internal forces. k_{bjk}, k_{Bjk} and x_{bj} $(j,\ k=1,\ 2)$ are bearing coefficients, nonlinear bearing coefficients and physical coordinates, respectively. The nonlinear coefficient of bearing δ as a small value is assumed to be expressed as ε, which can be perturbed as same as in component 1.

2.5 Formulation in assembling region

The perturbed equation for assembling region can be derived in accordance with substructures 1 and 2. Nonlinear Eq. (21) can be rewritten as a linearized equation by using perturbation approximation.

When the overall system is excited near the 1st natural frequency, there is a nonlinear vibration influence in substructure 2 as same as in the assembling region. It is regarded that the effect of nonlinearity of neighboring component is transimitted through the conjunction. Therefore, those nonlinearities are also subject to perturb. The conjunction region is assumed to be expressed as the linear combination of the component eigenvectors as follows:

$$\begin{aligned} \{x_{bj}\} = \{x_{bj}^{(0)}\} + \varepsilon\{x_{bj}^{(1)}\} \cong [\phi_{bj}]\{\{\xi_j^{(0)}\} \\ + \varepsilon\{\xi_j^{(1)}\}\} \quad (j=1,\ 2\} \end{aligned} \tag{22}$$

where $[\phi_{bj}]$ are eigenvectors of conjunction region, which is derived from the each component's eigenvectors corresponding to bearing parts. By substituting the internal force which is eliminated by conjunction and arranging with ε, the perturbed each component equation and conjunction region equations are obtained. Internal force of components 1 and 2 which are consist of bearing stiffness term is formulated as follows:

$$\begin{aligned} &\{f_{bj}\} = \{f_{bj}^{(0)}\} + \varepsilon \cdot \{f_{bj}^{(1)}\} \quad (j=1,\ 2) \\ &\{f_{b1}\} = [k_{b12}]\{x_{b2}^{(0)}\} + [k_{b11}]\{x_{b1}^{(0)}\} \\ &\quad + \varepsilon \cdot \{[k_{B11}]\{x_{b1}^{(1)}\} + [k_{B12}]\{x_{b2}^{(1)}\} \\ &\quad + [k_{b11}]\{x_{b1}^{(0)3}\} + [k_{b12}]\{x_{b2}^{(0)3}\}\} \\ &\{f_{b1}\} = -\{f_{b2}\} \end{aligned} \tag{23}$$

By substituting Eq. (23) into each component as internal force, the variables of each component ξ_{1i}, ξ_{2i}, are written to perturbation form. We obtain the sets of perturbed equations for components 1 and 2. Therefore, the equation of conjunction region is expressed with the 0th order and the 1st order of perturbation equations. The characteristic of this analysis method is to perturb the each nonlinear component in matrix form and perform the mode synthesis at bearing part.

3. Synthesis of Nonlinear System

In this section, a nonlinear rotor system, which is divided into three components; the nonlinear rotor, casing and nonlinear bearing as conjunction region as shown in Fig. 2, is considered. By truncating the vibration modes of the rotor and casing system according to the substructure synthesis method in elastic conjunction, which is applied to the unconstrained elastic mode synthesis in elastic conjunction, the reduced order equation of motion can be obtained.

3.1 Time domain response

The terms of the stiffness matrix of the interface are related to the stiffness matrix of the casing structure. The equations of motion of the typical linearized nonlinear rotor system and of the casing component of the generalized modal coordinates may be expressed as follows:

$$[M]\{\ddot{\xi}\}+[K]\{\xi\}=\{F(\xi,\ t)\} \tag{24}$$

where $[M]$, $[K]$ are the mass matrix and the stiffness matrix of the overall system which are composed of each component and assembling region.

$$\begin{aligned}\{\xi\}=&\{\{\xi_1^{(0)}\},\ \{\xi_1^{(1)}\},\ \{x_{b1}^{(0)}\},\ \{x_{b1}^{(1)}\},\ \{x_{b2}^{(0)}\},\\ &\{x_{b2}^{(1)}\},\ \{\xi_2^{(0)}\},\ \{\xi_2^{(1)}\}\}^T,\\ \{F(\xi,\ t)\}=&\{f_{\xi 1},\ f_{p1},\ -f_{b1}^{(0)},\ -f_{b1}^{(1)},\ f_{b2}^{(0)},\ f_{b2}^{(1)},\\ &f_{\xi 2},\ f_{p2}\}^T\end{aligned} \tag{25}$$

where the superscripts stand for perturbation order and subscripts stand for each component. In order to reduce the number of degrees of freedom, the following transformation is applied,

$$\{\xi\}=\begin{bmatrix}[I] & 0 & 0 & 0\\ 0 & [I] & 0 & 0\\ [\phi_{b1}] & 0 & 0 & 0\\ 0 & [\phi_{b1}] & 0 & 0\\ 0 & 0 & [\phi_{b2}] & 0\\ 0 & 0 & 0 & [\phi_{b2}]\\ 0 & 0 & [I] & 0\\ 0 & 0 & 0 & [T]\end{bmatrix}\begin{Bmatrix}\xi_1^{(0)}\\ \xi_1^{(1)}\\ \xi_2^{(0)}\\ \xi_2^{(1)}\end{Bmatrix}$$

$$=[P]\{\xi_j^{(l)}\},\ (j=1,\ 2),\ (l=0,\ 1) \tag{26}$$

where, $[I]$, $[P]$ are unit matrix and transformation matrix, respectively. By substituting Eq. (26) into Eq. (24) and pre-multiplying by $[P]^T$ yields

$$\{\ddot{\xi}_j^{(l)}\}+[K_t]\{\xi_j^{(l)}\}=\{f_\eta(\xi^{(l)},\ t)\} \tag{27}$$

$$\{f_\eta(\xi^{(l)},\ t)\}$$

$$=\begin{Bmatrix}[\phi_{a1}]^T\cdot\{f_{\xi 1i}\}\\ [\phi_{a1}]^T\cdot\{f_{p1}(\xi_{1i}^{(0)},\ \dot{\xi}_{1i}^{(0)})\}+[\phi_{b1}]^T\cdot\{f_{b1}^{(1)}\}\\ [\phi_{a2}]^T\cdot\{f_{\xi 2i}\}\\ [\phi_{a2}]^T\cdot\{f_{p2}(\xi_{2i}^{(0)})\}+[\phi_{b2}]^T\cdot\{f_{b2}^{(1)}\}\end{Bmatrix}$$

$$=\begin{Bmatrix}\{f_{\eta 1}^{(0)}\}\\ \{f_{\eta 1}^{(1)}\}\\ \{f_{\eta 2}^{(0)}\}\\ \{f_{\eta 2}^{(1)}\}\end{Bmatrix}$$

where $[\phi_{a1}]$, $[\phi_{a2}]$ are eigenvectors of only internal region in each substructures. $[K_t]$ is reduced order stiffness matrix term. The number of equation is depend on the number of adopted modes in Eq. (23) and the number of adopted terms in Eq. (21). This point is the advantage in case of using the substructure synthesis method. After obtaining the reduced order equation, those equations are divided into two sub-equations by the terms of perturbation order.

perturbation 0th order part:

$$\begin{Bmatrix}\{\ddot{\xi}_1^{(0)}\}\\ \{\ddot{\xi}_2^{(0)}\}\end{Bmatrix}+\begin{bmatrix}[\omega_{1i}^2]+[a_1] & [a_2]\\ [a_3] & [\omega_{2i}^2]+[a_4]\end{bmatrix}\begin{Bmatrix}\{\xi_1^{(0)}\}\\ \{\xi_2^{(0)}\}\end{Bmatrix}=\begin{Bmatrix}\{f_{\eta 1}^{(0)}\}\\ \{f_{\eta 2}^{(0)}\}\end{Bmatrix} \tag{28}$$

$[a_1]=[\phi_{b1}]^T[k_{b11}][\phi_{b1}]$, $[a_2]=[\phi_{b1}]^T[k_{b12}][\phi_{b2}]$, $[a_3]=[\phi_{b2}]^T[k_{b21}][\phi_{b1}]$, $[a_4]=[\phi_{b2}]^T[k_{b22}][\phi_{b2}]$

perturbation 1st order part:

$$\begin{Bmatrix}\{\ddot{\xi}_1^{(1)}\}\\ \{\ddot{\xi}_2^{(1)}\}\end{Bmatrix}+\begin{bmatrix}[\omega_{1i}^2]+[b_1] & [b_2]\\ [b_3] & [\omega_{2i}^2]+[b_4]\end{bmatrix}\begin{Bmatrix}\{\xi_1^{(1)}\}\\ \{\xi_2^{(1)}\}\end{Bmatrix}=\begin{Bmatrix}\{f_{\eta 1}^{(1)}\}\\ \{f_{\eta 2}^{(1)}\}\end{Bmatrix} \tag{29}$$

$[b_1]=[\phi_{b1}]^T[k_{B11}][\phi_{b1}]$, $[b_2]=[\phi_{b1}]^T[k_{B12}][\phi_{b2}]$, $[b_3]=[\phi_{b2}]^T[k_{B21}][\phi_{b1}]$, $[a_4]=[\phi_{b2}]^T[k_{B22}][\phi_{b2}]$

where the $\{f_{\eta 1,2}^{(l)}\}$ $(l=0,\ 1)$ are displacement dependent outforce vector after the elimination of the internal force. After obtaining the responses of the overall system, then those are changed into the physical coordinates at last.

3.2 Frequency domain response

Here, an analytical frequency domain technique based on the modal superposition principle is presented in the context of obtaining the fundamental harmonic response of the rotor system with nonlinearities. This method, which is being used as an alternative to numerical integration procedures for steady state periodic response, is compatible with the need of the modal analysis to identify the frequency response characteristic of MDOF systems with nonlinearities. The equation of motion of a nonlinear rotor system modeled as an n degrees of freedom discrete system in the modal coordinates can be expressed in the form of each substructure as Eq. (11) and Eq. (18).

The overall system is analyzed by combining subsystems with conjunction part as shown in Fig. 1 schematically. The internal forces can be eliminated by the synthesis as in time domain analysis. We seek the steady-state harmonic solution by the iteration, which is a process successive approximation. Assuming the first approximation to be $\xi_0 = A\cos\omega t + B\sin\omega t$ and its substitution into the differential equation results in a polynomial expression of trigonometry. By equating the coefficients of $\cos\omega t$ and $\sin\omega t$, and squaring these results, the relationship between the frequency, amplitude and force terms are obtained. This procedure can be repeated any number of times to achieve the desired accuracy.

4. Numerical Examples

The rotor system whose properties are tabulated in Table 1 is considered in this analysis with cubic stiffness type nonlinearity in shaft and bearing of rotor system. The material coefficient γ in Eq. (1) is 0.1. The perturbation parameters ε in Eq. (11) and δ in Eq. (20) are set as follows:

$$\varepsilon = 0.09,\ \delta = 0.1$$

The rotor and casing are to be analyzed as uniform beams modeled by the twenty finite elements, respectively. The material density and

Table 1 Properties of the rotor system.

Length of rotor & casing	L(mm)	800
Diameter of rotor	D_R(mm)	16
Diameter of casing	D_P(mm)	50
Young's modulus of rotor	E(N/m^2)	2.1×10^{11}
Young's modulus of casing	E(N/m^2)	2.1×10^{11}
Density of shaft & casing	ρ(kg/m^3)	7.81×10^{3}
Bearing coefficient	k_{b11}, k_{b22}(N/m)	1.0×10^{6}
Nonlinear Bearing coefficient	k_{B11}, k_{B22}(N/m)	1.0×10^{6}

Table 2 Natural frequency of rotor system (Hz)

Mode No.	FEM (DOF=168)	SSM Adopted mode number (Shaft/Casing element)		
		10/10	20/20	40/40
1	73.29	73.10	73.28	73.29
2	164.26	162.52	164.23	164.27
3	310.64	312.13	310.62	310.63
4	488.22	487.27	488.23	488.23
5	668.24	668.54	668.81	668.83

the modules of elasticity of rotor and casing are assumed to be identical, respectively. The damping ratio of the rotor and casing in each normal mode is given by $\zeta = 0.01$. The casing is constrained to foundation. The damping of the bearing and constraint is ignored.

Table 2 shows the natural frequency of rotor system in Hz by FEM and SSM (Substructure Synthesis Method) where the adopted component modes are changed. The results obtained by considering 20 modes of each component in SSM are in very good agreement with those obtained by FEM. Therefore, 20 modes of each component are considered for the further response analysis.

Figure 3 shows the critical speeds for lower three modes of rotor system, which are calculated by FEM and SSM. The first three critical speeds in SSM that are in good agreement with those obtained by FEM varying typically with support stiffness.

In this analysis model, the response is considered at the representative nodal point (x-direction) of rotor system such as at the middle of shaft (unbalancing point).

Figure 4 compares time domain responses of SSM with those of FEM when the center of rotor is excited at 451 rad/s by unbalance of rotor (unbalance=88.3 g·mm) where the first natural frequency of the system is 460.52 rad/s. It can be observed at the selected point that by using only 20 modes relatively accurate responses of the rotor system can be simulated comparing with responses of FEM, as shown in Fig. 4 (a), (b)

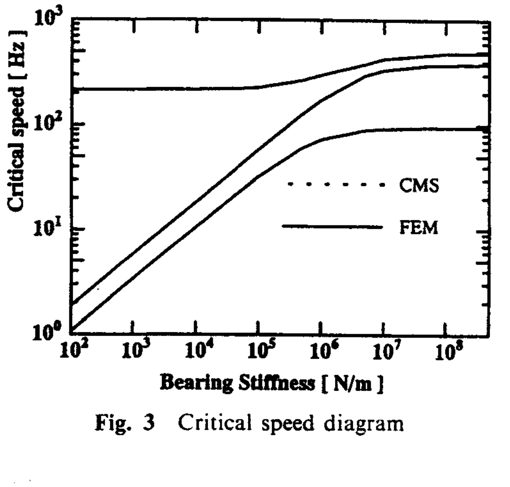

Fig. 3 Critical speed diagram

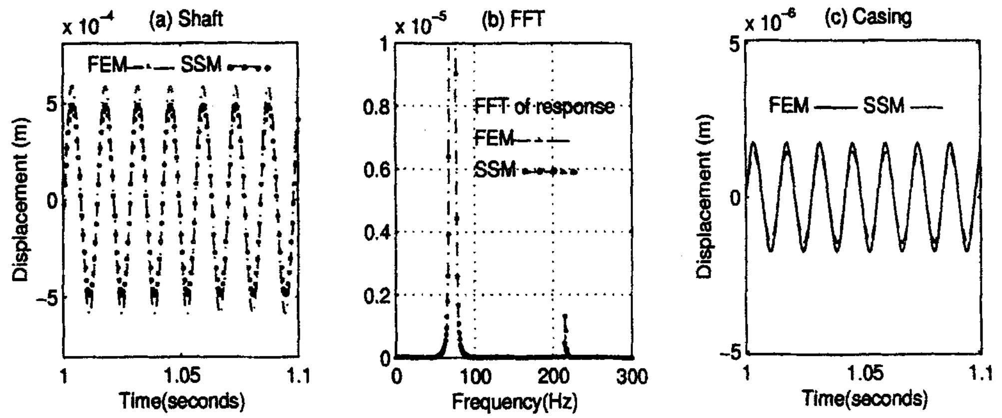

Fig. 4 Responses of FEM and SSM at shaft and casing.

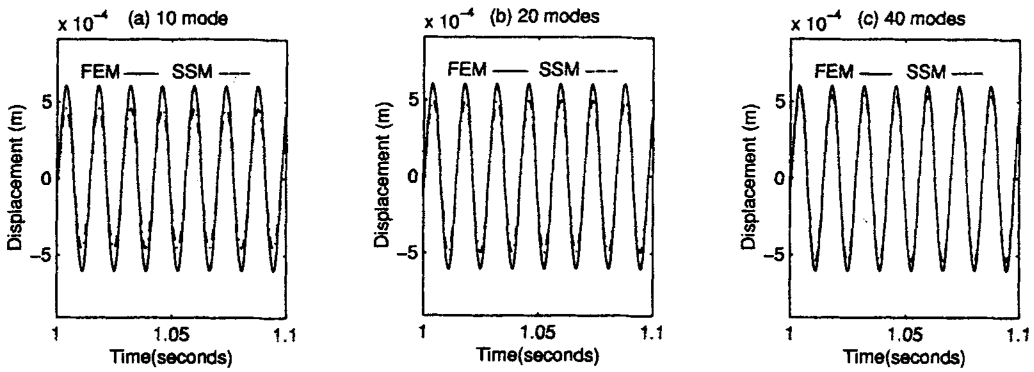

Fig. 5 Responses of FEM and SSM with changing modes.

and (c). The difference of the responses is regarded as the influence of discarding the higher modes of the system in SSM with adopting only 20 modes and the influence of working in the higher frequencies in FEM result. Next, we show the computing time to verify the effectiveness the proposed method in time domain analysis based on one frequency by FEM and SSM. As an example, the calculation time of the responses of Fig. 4(a) is regarded. The proposed method takes 5 minutes 8 seconds to calculate the time responses until 1.1 seconds, while direct integration takes 39 minutes 40 seconds by using the computer *Ultra 1, SunMicrosystem Co.*. As a result, it can be observed in this study that drastic reduction in computational time can be obtained while keeping the accuracy by adopting only lower 20 modes what is a critical factor in the analysis of structural dynamics with a large number of degree of freedom.

Figure 5 compares time domain responses of SSM with FEM in the same condition with Fig. 4 by changing the adopting modes. It can be observed at the selected point that by using 10, 20, 40 modes relatively accurate responses of the rotor system can be simulated comparing with response of FEM, as shown in Fig. 5 (a), (b) and (c). As can be noticed from Fig. 5(c), there is not a notable response difference between FEM and SSM responses. The proposed method takes 7 minutes 23 seconds to calculate the time response until 1.1 seconds, while direct integration takes 39 minutes 40 seconds, as shown in Fig. 5(c).

Figure 6 shows the corresponding responses at the middle of shaft and their FFT for the case where the rotor and bearing nonlinear coefficients are $\gamma=0.1$ and $\delta=0.1$ in physical coordinates, respectively. The responses for the case of nonlinearity in shaft only and in bearing only for the same excitation frequency are shown in order to compare their effect on the overall system. It can be observed that the FFT result of the nonlinear bearing response have higher frequency component (*perturbation 1st order amplitude* $=8.08\times10^{-7}$(m)) than those for the nonlinear response of shaft component (*perturbation 1st order amplitude* $=7.29\times10^{-8}$(m)) which is the perturbation 1st order amplitude in terms of the nonlinearity characteristic, as shown in Fig. 6 (a) and (b).

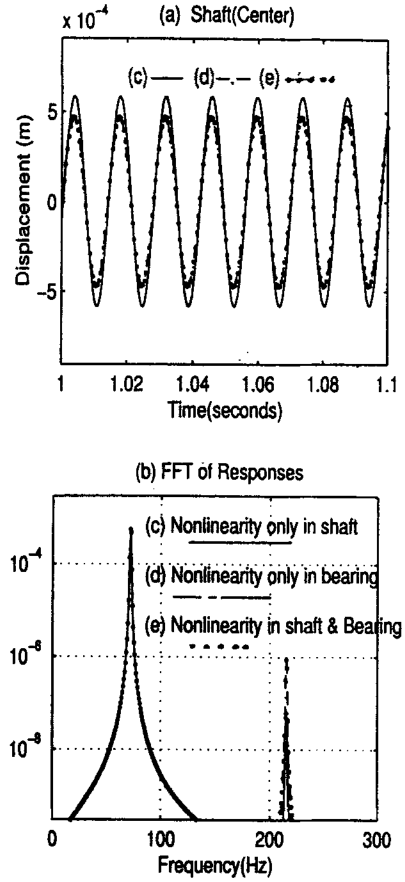

Fig. 6 Nonlinear responses and its FFT of the system.

Figure 7 shows the sensitivity of bearing nonlinearity and shaft nonlinearity to the overall system by changing the nonlinear parameter in physical coordinates from 0.01 to 0.2 with nearly same linear stiffness (1.0×10^6(N/m)).

Fig. 7(a) is a FFT result of time response of at the middle of shaft where the value of ($X_1=[\Phi_1]\{\xi_1^{(0)}+\varepsilon\xi_1^{(1)}\}$, $X_1^{(1)}=\varepsilon[\Phi_1]\{\xi_1^{(1)}\}$) are taken from the Eqs. (28) and (29). Investigation of the

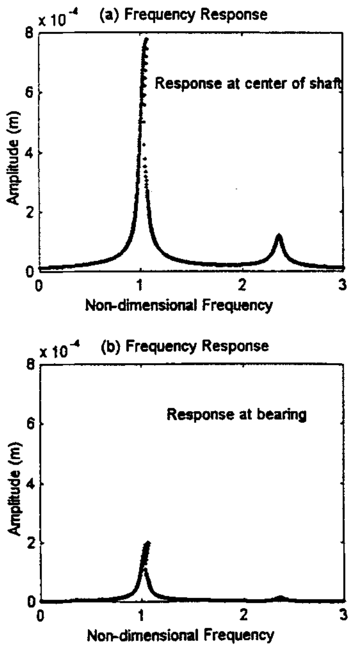

Fig. 8 Frequency response

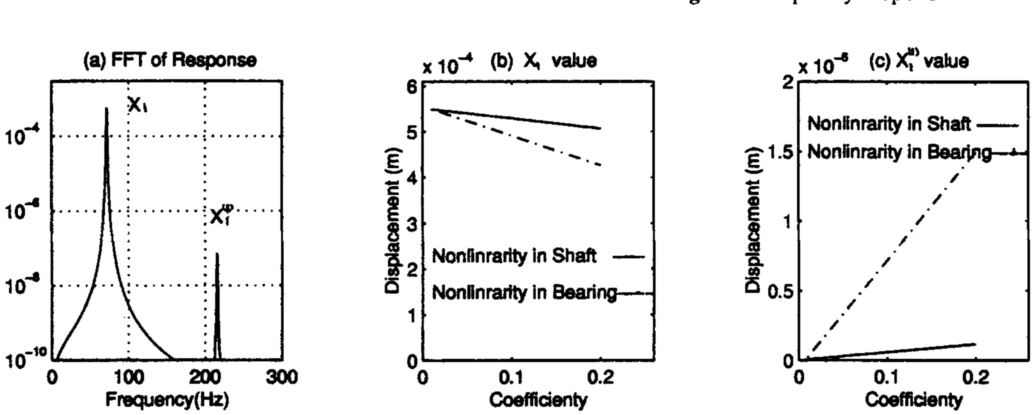

Fig. 7 Nonlinearity sensitivity

sensitivity, as shown in Fig. 7 (b) and (c), reveals that the sensitivity of the bearing is bigger than that of the shaft. It is regarded that the nonlinearity in the bearing had a more significant effect on the rotor's response compared with the nonlinearity in the shaft system.

Figure 8 shows the nonlinear frequency response at 6 cm from the middle of rotor and at the bearing as a function of the frequency in non-dimensional form, which is equal to the rotating speed divided by the first critical speed. It can be observed that the resonance peak of each selected point is not straight, but slopes to the right for hardening nonlinear system, as shown in Fig. 8 (a) and (b).

5. Conclusions

In this paper, the vibration analysis method of a rotor-bearing and casing nonlinear system that represents a large mechanical nonlinear structure were theoretically formulated, using the perturbation method and the substructure synthesis method. By applying the perturbation method, the approximated solutions were obtained in the nonlinear component and assembling region. The whole governing equations were synthesized using the substructure synthesis method and carried out the nonlinear formulation. It was shown that nonlinear responses could be efficiently calculated by selecting a proper number of vibration modes for economical calculation. It was observed that the nonlinearity in the bearing had a more significant effect on the rotor's response compared with the nonlinearity in the shaft system.

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